

Chapter 6

Fracture Analysis of Functionally Graded Materials (FGMs)

6.1 INTRODUCTION

6.2 ANALYTICAL SOLUTION FOR NEAR CRACK TIP

Details of anisotropic elasticity, as discussed in Section 4.2.1, are also valid for inhomogeneous anisotropic materials. In this section, analytical solutions for near a crack tip in functionally graded materials are discussed.

6.2.1 Average material properties

In order to better describe and compare the behaviour of functionally graded materials, four independent average material properties, which include stiffness parameter E , Poisson's ratio ν , stiffness ratio δ^4 , and shear modulus κ_0 , can be defined. For the plane stress state they are defined as:

$$E = \sqrt{E_{11}E_{22}} \quad (6.1)$$

$$\nu = \sqrt{\nu_{12}\nu_{21}} \quad (6.2)$$

$$\delta^4 = \frac{E_{11}}{E_{22}} = \frac{\nu_{12}}{\nu_{21}} \quad (6.3)$$

$$\kappa_0 = \frac{E}{2G_{12}} - \nu \quad (6.4)$$

and for a plane strain problem,

$$E = \left[\frac{E_{11}E_{22}}{(1-\nu_{13}\nu_{31})(1-\nu_{23}\nu_{32})} \right]^{\frac{1}{2}} \quad (6.5)$$

$$\nu = \sqrt{\frac{(\nu_{12} + \nu_{13}\nu_{32})(\nu_{21} + \nu_{23}\nu_{31})}{(1-\nu_{13}\nu_{31})(1-\nu_{23}\nu_{32})}} \quad (6.6)$$

$$\delta^4 = \frac{E_{11}(1-\nu_{23}\nu_{32})}{E_{22}(1-\nu_{13}\nu_{31})} \quad (6.7)$$

$$\kappa_0 = \frac{E}{2G_{12}} - \nu \quad (6.8)$$

6.2.2 Mode I near tip fields in FGM composites

There is very limited available literature on the analytical solution for fracture analysis in FGM composites. They are usually derived in the form of infinite series, which cannot be transformed into an appropriate form to derive explicit enrichment functions. Here, the solution for mode I cracks, originally derived by Ozturk and Erdogan (1997), is briefly reviewed.

Consider a typical crack within a linear elastic FGM composite, with orthotropic material axes x_1 and x_2 , as depicted in Fig. 6.1. The crack is positioned on the $x_2 = 0$ line within the $-a < x < a$ interval. So, the crack is located in a plane perpendicular to the direction of property gradient and the principal axes of orthotropy are parallel and perpendicular to the crack.

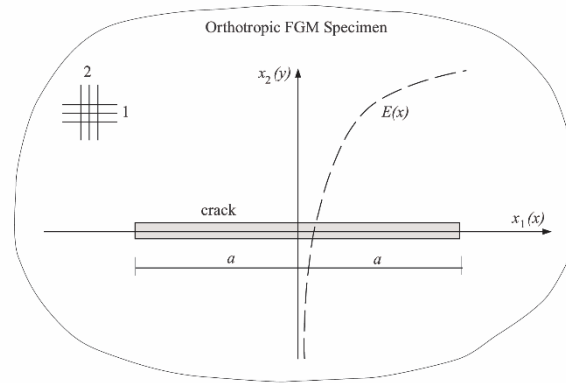


Figure 6.1 Definitions for an ideal mode I crack in an FGM material.

The average material properties of Section 6.2.1 are, in general, functions of x_1 and x_2 . Nevertheless, the following simplifications which are in line with usual industrial applications are assumed:

- Variations are assumed to be only in x_1 direction.

- Variations of the Poisson's coefficient are neglected. This is confirmed by a number of previous studies, such as Delale and Erdogan (1983) and Erdogan and Wu (1993), which will be examined by numerical simulations.
- Variations of E_{11} , E_{22} and G_{12} are assumed to be proportionally increasing.

As a result, only the elasticity modulus remains a function of x_1 and the remaining coefficients are assumed to be constant.

It should be mentioned that the special case of $\kappa_0=1$, $\delta=1$ represents an isotropic material. For a homogenous orthotropic material, $-1 < \kappa_0 < \infty$, and it can be proved that no solution is obtained if $\kappa_0 < -1$. The same conclusions can be made for non-homogenous materials.

New coordinates are defined from the scaling factor δ ,

$$x = \frac{x_1}{\sqrt{\delta}} \quad (6.9)$$

$$y = \sqrt{\delta} x_2 \quad (6.10)$$

Similarly, displacement and stress components are scaled:

$$u_x(x, y) = u_1(x_1, x_2) \sqrt{\delta} \quad (6.11)$$

$$u_y(x, y) = \frac{u_2(x_1, x_2)}{\sqrt{\delta}} \quad (6.12)$$

$$\sigma_{xx}(x, y) = \frac{1}{\delta} \sigma_{11}(x_1, x_2) \quad (6.13)$$

$$\sigma_{yy}(x, y) = \delta \sigma_{22}(x_1, x_2) \quad (6.14)$$

$$\sigma_{xy}(x, y) = \sigma_{12}(x_1, x_2) \quad (6.15)$$

Stress-displacement relations can now be written as,

$$\sigma_{xx}(x, y) = \frac{\bar{E}(x, y)}{1 - \nu^2} \left(\frac{\partial u_x(x, y)}{\partial x} + \nu \frac{\partial u_y(x, y)}{\partial y} \right) \quad (6.16)$$

$$\sigma_{yy}(x, y) = \frac{\bar{E}(x, y)}{1 - \nu^2} \left(\frac{\partial u_y(x, y)}{\partial y} + \nu \frac{\partial u_x(x, y)}{\partial x} \right) \quad (6.17)$$

$$\sigma_{xx}(x, y) = \frac{\bar{E}(x, y)}{2(\kappa_0 + \nu)} \left(\frac{\partial u_x(x, y)}{\partial y} + \frac{\partial u_y(x, y)}{\partial x} \right) \quad (6.18)$$

where

$$\bar{E}(x, y) = E(x_1, x_2) \quad (6.19)$$

The equilibrium equations are then transformed to:

$$\frac{\partial^2 u_x}{\partial y^2} + \beta_1 \frac{\partial^2 u_x}{\partial x^2} + \beta_2 \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\beta_1}{E} \frac{\partial \bar{E}}{\partial x} \left(\frac{\partial u_x}{\partial x} + \nu \frac{\partial u_y}{\partial y} \right) + \frac{1}{E} \frac{\partial \bar{E}}{\partial y} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) = 0 \quad (6.20)$$

$$\frac{\partial^2 u_y}{\partial x^2} + \beta_1 \frac{\partial^2 u_y}{\partial y^2} + \beta_2 \frac{\partial^2 u_x}{\partial x \partial y} + \frac{\beta_1}{E} \frac{\partial \bar{E}}{\partial y} \left(\frac{\partial u_y}{\partial y} + \nu \frac{\partial u_x}{\partial x} \right) + \frac{1}{E} \frac{\partial \bar{E}}{\partial x} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) = 0 \quad (6.21)$$

with

$$\beta_1 = \frac{2(\kappa_0 + \nu)}{1 - \nu^2}, \quad \beta_2 = 1 + \nu \beta_1 \quad (6.22)$$

$E(x, y)$ is assumed to be a monotonously increasing function of x_1 as:

$$E(x_1, x_2) = E(x_1) = E_0 e^{\alpha_1 x_1} = \bar{E}(x) = E_0 e^{\gamma x} \quad (6.23)$$

with

$$\gamma = \alpha_1 \sqrt{\delta} \quad (6.24)$$

and the equilibrium equations become

$$\frac{\partial^2 u_x}{\partial y^2} + \beta_1 \frac{\partial^2 u_x}{\partial x^2} + \beta_2 \frac{\partial^2 u_y}{\partial x \partial y} + \beta_1 \gamma \left(\frac{\partial u_x}{\partial x} + \nu \frac{\partial u_y}{\partial y} \right) = 0 \quad (6.25)$$

$$\frac{\partial^2 u_y}{\partial x^2} + \beta_1 \frac{\partial^2 u_y}{\partial y^2} + \beta_2 \frac{\partial^2 u_x}{\partial x \partial y} + \gamma \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) = 0 \quad (6.26)$$

A homogeneous differential equation is then obtained if $\gamma = 0$.

Following the formulation of Ozturk and Erdogan (1997), the solutions are assumed to be in the following general forms (due to symmetry, only $x_2 > 0$ region is considered):

$$u_x(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 b_j(k) B_j(k) e^{\lambda_j y - i k x} dk \quad (6.27)$$

$$u_y(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 B_j(k) e^{\lambda_j y - i k x} dk \quad 0 < y < \infty \quad (6.28)$$

$$\sigma_{xx}(x, y) = \frac{\bar{E}(x)}{1-\nu^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 (\nu\lambda_j - ib_j k) B_j e^{\lambda_j y - ikx} dk \quad (6.29)$$

$$\sigma_{yy}(x, y) = \frac{\bar{E}(x)}{1-\nu^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 (\lambda_j - i\nu b_j k) B_j e^{\lambda_j y - ikx} dk \quad (6.30)$$

$$\sigma_{xy}(x, y) = \frac{\bar{E}(x)}{2(\kappa_0 + \nu)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 (\lambda_j b_j - ik) B_j e^{\lambda_j y - ikx} dk, \quad 0 < y < \infty \quad (6.31)$$

where λ_i are the roots of the characteristic equation

$$\lambda^4 - (\nu\gamma^2 + 2(k^2 + i\gamma k)\kappa_0)\lambda^2 + (k^2 + i\gamma k)^2 = 0 \quad (6.32)$$

or more explicitly,

$$\lambda_1 = -\lambda_3 = \left\{ \frac{1}{2} \left(\omega^2 + 2\kappa_0 \eta \right) + \frac{1}{2} \sqrt{\left(\omega^2 + 2\kappa_0 \eta \right)^2 - 4\eta^2} \right\}^{\frac{1}{2}} \quad (6.33)$$

$$\lambda_2 = -\lambda_4 = \left\{ \frac{1}{2} \left(\omega^2 + 2\kappa_0 \eta \right) - \frac{1}{2} \sqrt{\left(\omega^2 + 2\kappa_0 \eta \right)^2 - 4\eta^2} \right\}^{\frac{1}{2}} \quad (6.34)$$

with

$$\omega^2 = \nu\gamma^2, \quad \eta = k^2 + i\gamma k \quad (6.35)$$

and

$$b_j(k) = \frac{(i\beta_2 k - \gamma\beta_1 \nu)}{\lambda_j^2 - \beta_1 \eta} \lambda_j \quad (6.36)$$

$B_3(k)$ and $B_4(k)$ should be determined from the boundary conditions. After lengthy calculations, the following set of simultaneous equations is obtained,

$$\sum_{j=3}^4 (\lambda_j b_j - ik) B_j = 0 \quad (6.37)$$

$$(-ik)(B_3 + B_4) = \int_{-\bar{a}}^{\bar{a}} \frac{\partial u_y(s, 0^+)}{\partial s} e^{iks} ds \quad (6.38)$$

where

$$\bar{a} = \frac{a}{\sqrt{\delta}} \quad (6.39)$$

The final solutions of the displacement field of the crack face u_2 and the stress intensity factors K_I at the two crack tips are (Ozturk and Erdogan, 1997),

$$\frac{u_2(x_1, 0^+)}{\omega_0} = -\sqrt{1-r^2} \left[\sum_{n=1}^{\infty} \frac{A_n}{n} U_{n-1}(r) \right] \quad (6.40)$$

$$\frac{K_I(a)}{K_0} = -e^{a_1 a} \sum_{n=1}^{\infty} A_n \quad (6.41)$$

$$\frac{K_I(-a)}{K_0} = e^{-a_1 a} \sum_{n=1}^{\infty} (-1)^n A_n \quad (6.42)$$

where p_0 is a constant corresponding to the amplitude of the external loading and

$$K_0 = p_0 \sqrt{a} \quad (6.43)$$

$$r = x_1 / a \quad (6.44)$$

$$\omega_0 = \frac{(s_1 + s_2) \delta p_0 a}{E_0} \quad (6.45)$$

$$\kappa_1 = \sqrt{\kappa_0^2 - 1}, \quad s_1 = \sqrt{\kappa_0 + \kappa_1}, \quad s_2 = \sqrt{\kappa_0 - \kappa_1} \quad (6.46)$$

and A_n and U_n are determined from

$$\sum_{n=1}^{\infty} A_n [U_{n-1}(r) + R_n(r)] = e^{-a_1 a r} f(r) \quad (6.47)$$

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad t = \cos \theta = t_1 / a, \quad s = t_1 / \sqrt{\delta} \quad (6.48)$$

where

$$R_n(r) = \frac{1}{\pi} \int_{-1}^1 \frac{a \cos(n\theta) M_1(ra, ta)}{\sqrt{1-t^2}} dt \quad (6.49)$$

$$f(r) = \frac{p(x_1)}{p_0} \quad (6.50)$$

$p(x_1)$ is the crack boundary conditions ($\sigma_{22}(x_1, 0^+) = p(x_1)$, $-a < x_1 < a$) and

$$M_1(x_1, t_1) = \frac{1}{2\sqrt{\delta}} L_1\left(\frac{x_1}{\sqrt{\delta}}, \frac{t_1}{\sqrt{\delta}}\right) = \frac{1}{2\sqrt{\delta}} L_1(x, s) \quad (6.51)$$

where L_1 is the Fredholm kernel,

$$L_1(x, s) = \int_0^\infty 2 \{ \operatorname{Re}(h(k)) \cos(k(s-x)) - \operatorname{Im}(h(k)) \sin(k(s-x)) \} dk \quad (6.52)$$

with

$$h(k) = \begin{cases} -i \left[\frac{(s_1 + s_2)(k + i\gamma)}{\lambda_1 + \lambda_2} - 1 \right] & \kappa_0 \neq 1 \\ -i \left[\frac{2(k + i\gamma)}{\sqrt{\nu\gamma^2 + 4(k^2 + i\gamma k)}} - 1 \right] & \kappa_0 = 1 \end{cases} \quad (6.53)$$

In real materials, $s_1 + s_2$ is a real number. For inhomogeneous isotropic materials with a constant Poisson's ratio:

$$\delta^4 = \frac{E_{11}}{E_{22}} = 1, \kappa_0 = 1, \bar{a} = a, x = x_1, s_1 = 1, s_2 = 1 \quad (6.54)$$

the following simplified Fredholm kernel is obtained,

$$L_1(x, s) = \gamma e^\xi \left\{ \nu_0 \frac{|\xi|}{\xi} K_1(\nu_0 |\xi|) + K_0(\nu_0 |\xi|) \right\} - \frac{2}{s-x} \quad (6.55)$$

where $K_0(z)$ and $K_1(z)$ are the modified Bessel functions (Clarke and Hess, 1978 and Watson, 1995),

$$K_\alpha(z) = \int_0^\infty e^{-z \cosh t} \cosh(\alpha t) dt, \alpha = 1, 2 \quad (6.56)$$

and

$$\xi = \frac{\gamma}{2}(s-x) = \frac{\alpha_1}{2}(t_1 - x_1) \quad (6.57)$$

$$\nu_0 = \begin{cases} \sqrt{1+\nu} & \text{plane stress} \\ \sqrt{\frac{1}{1-\nu}} & \text{plane strain} \end{cases} \quad (6.58)$$

functions are calculated at the crack tip. Consequently, the following replacement is adopted in Section 4.3.1 for FGM materials

$$\chi_k \rightarrow \chi_k^{tip}, \quad \chi_k = c_{ij}, p_k, q_k, s_k \quad (6.60)$$

The displacement fields u_1 and u_2 , in x and y directions, respectively, are defined as,

$$\begin{aligned} u_1 = & K_I \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{1}{s_1^{tip} - s_2^{tip}} \left[s_1^{tip} p_2^{tip} g_2(\theta) - s_2^{tip} p_1^{tip} g_1(\theta) \right] \right\} + \\ & K_{II} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{1}{s_1^{tip} - s_2^{tip}} \left[p_2^{tip} g_2(\theta) - p_1^{tip} g_1(\theta) \right] \right\} \end{aligned} \quad (6.61)$$

$$\begin{aligned} u_2 = & K_I \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{1}{s_1^{tip} - s_2^{tip}} \left[s_1^{tip} q_2^{tip} g_2(\theta) - s_2^{tip} q_1^{tip} g_1(\theta) \right] \right\} + \\ & K_{II} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{1}{s_1^{tip} - s_2^{tip}} \left[q_2^{tip} g_2(\theta) - q_1^{tip} g_1(\theta) \right] \right\} \end{aligned} \quad (6.62)$$

where

$$g_i(\theta) = \sqrt{\cos(\theta) + s_i^{tip} \sin(\theta)} \quad (i=1,2) \quad (6.63)$$

$$p_k^{tip} = c_{11}^{tip} (s_k^{tip})^2 + c_{12}^{tip} - c_{16}^{tip} s_k^{tip} \quad (k=1,2) \quad (6.64)$$

$$q_k^{tip} = c_{12}^{tip} s_k^{tip} + \frac{c_{22}^{tip}}{s_k^{tip}} - c_{26}^{tip} \quad (k=1,2) \quad (6.65)$$

c_{ij}^{tip} are the compliance components defined in (4.8) and s_k^{tip} are the roots of the characteristic equation (4.23), all evaluated at the crack tip position. Also, the asymptotic stress components are written as

$$\begin{aligned} \sigma_{11} = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{s_1^{tip} s_2^{tip}}{s_1^{tip} - s_2^{tip}} \left[\frac{s_2^{tip}}{g_2(\theta)} - \frac{s_1^{tip}}{g_1(\theta)} \right] \right\} + \\ & \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{1}{s_1^{tip} - s_2^{tip}} \left[\frac{(s_2^{tip})^2}{g_2(\theta)} - \frac{(s_1^{tip})^2}{g_1(\theta)} \right] \right\} \end{aligned} \quad (6.66)$$

$$\begin{aligned} \sigma_{22} = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{1}{s_1^{tip} - s_2^{tip}} \left[\frac{s_1^{tip}}{g_2(\theta)} - \frac{s_2^{tip}}{g_1(\theta)} \right] \right\} + \\ & \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{1}{s_1^{tip} - s_2^{tip}} \left[\frac{1}{g_2(\theta)} - \frac{1}{g_1(\theta)} \right] \right\} \end{aligned} \quad (6.67)$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{s_1^{tip} s_2^{tip}}{s_1^{tip} - s_2^{tip}} \left[\frac{1}{g_1(\theta)} - \frac{1}{g_2(\theta)} \right] \right\} + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{1}{s_1^{tip} - s_2^{tip}} \left[\frac{s_1^{tip}}{g_1(\theta)} - \frac{s_2^{tip}}{g_2(\theta)} \right] \right\} \quad (6.68)$$

6.3 STRESS INTENSITY FACTOR

6.3.1 J integral

The J integral has been frequently used to compute the stress intensity factors in various mixed mode fracture problems. The general form of path independent J integral can be written as (2.153),

$$J = \int_{\Gamma} \left(w_s \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) n_j d\Gamma \quad (6.69)$$

where Γ is an arbitrary contour surrounding the crack tip, n_j is the j^{th} component of the outward unit normal to Γ , δ_{ij} is the Kronecker delta and the Cartesian coordinate system whose x axis is parallel to the crack surface is considered. w_s is the strain energy density, defined in (2.154).

Using the equivalent domain integral formulation, Eq. (6.69) for inhomogeneous materials is transformed into (Kim and Paulino, 2003a),

$$J = \int_A (\sigma_{ij} u_{i,1} - w_s \delta_{1j}) q_{,j} dA + \int_A (\sigma_{ij} u_{i,1} - w_s \delta_{1j})_{,j} q dA \quad (6.70)$$

where q is a smooth function varying from $q=1$ on the interior boundary of A to $q=0$ on the outer one.

6.3.2 Interaction integral

The interaction or M integral is a powerful and accurate approach, based on superimposition of auxiliary and actual fields, for evaluation of stress intensity factors. Yau *et al.* (1980) used the M integral to evaluate the stress intensity factors in isotropic homogeneous materials. Wang *et al.* (1980) extended the method to orthotropic homogeneous materials. Then, Yau (1979) used the same approach to compute SIFs at the interface of two materials.

The classical form of the interaction integral for computation of mode I and II stress intensity factors are based on the superposition of auxiliary and actual fields,

$$J^s = J + J^{aux} + M^I \quad (6.71)$$

where J and J^{aux} are the J integrals corresponding to actual and auxiliary fields, respectively. J^{aux} is defined by

$$J^{aux} = \int_A \left(\sigma_{ij}^{aux} u_{i,1}^{aux} - w_s^{aux} \delta_{1j} \right) q_{,j} dA + \int_A \left(\sigma_{ij}^{aux} u_{i,1}^{aux} - w_s^{aux} \delta_{1j} \right)_{,j} q dA \quad (6.72)$$

and M^I is the local interaction integral calculated by

$$M^I = \int_A \left\{ \sigma_{ij}^{aux} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} - \frac{1}{2} \left(\sigma_{ik} \varepsilon_{ik}^{aux} + \sigma_{ik}^{aux} \varepsilon_{ik} \right) \delta_{1j} \right\} q_{,j} dA + \int_A \left\{ \sigma_{ij}^{aux} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} - \frac{1}{2} \left(\sigma_{ik} \varepsilon_{ik}^{aux} + \sigma_{ik}^{aux} \varepsilon_{ik} \right) \delta_{1j} \right\}_{,j} q dA \quad (6.73)$$

The energy release rate in elastic media is calculated as

$$G = J = t_{11} K_I^2 + t_{12} K_I K_{II} + t_{22} K_{II}^2 \quad (6.74)$$

where t_{ij} are defined in (4.83)-(4.85). Using the same methodology discussed in Section 4.4.3 leads to the following linear algebraic equations to calculate actual mode I and II stress intensity factors from the local interaction integral M^I ,

$$M^I \left(K_I^{aux} = 1, K_{II}^{aux} = 0 \right) = 2t_{11} K_I + t_{12} K_{II} \quad (6.75)$$

$$M^I \left(K_I^{aux} = 0, K_{II}^{aux} = 1 \right) = t_{12} K_I + 2t_{22} K_{II} \quad (6.76)$$

6.3.3 FGM auxiliary fields

Most of the available research for FGMs are based on the assumption of isotropic FGMs. They include XFEM by Dolbow and Gosz (2002), EFG by Rao and Rahman (2003) and FEM by Kim and Paulino (2003a, 2003b), among the others. They are either based on incompatibility or constant constitutive tensor formulations. According to discussions of Dolbow and Gosz (2002) for isotropic FGMs and Kim and Paulino (2003c) for orthotropic FGMs, the asymptotic crack tip fields in homogeneous orthotropic media can be used as the auxiliary fields for computation of SIFs in orthotropic FGM problems.

The governing equations for an elasticity problem are equilibrium equations, consistency conditions, and the constitutive relations. Unfortunately, if the analytical asymptotic fields for orthotropic materials are used for definition of auxiliary fields in fracture analysis of FGM composites, one of the main three governing equations may not be satisfied.

Based on which set of equations are ignored in defining the auxiliary fields, three different methods have been proposed: Non-equilibrium formulation, incompatibility formulation and constant-constitutive-tensor formulation.

The first method only satisfies the constitutive and compatibility equations

$$\sigma_{ij}^{aux} = D_{ijkl}(x) \varepsilon_{kl}^{aux}, \varepsilon_{ij}^{aux} = \frac{1}{2} (u_{i,j}^{aux} + u_{j,i}^{aux}), \sigma_{ij,j}^{aux} \neq 0 \quad (6.77)$$

where D_{ijkl} are the components of the material elasticity tensor, defined in (4.3). In the second approach, only equilibrium and the constitutive equations are satisfied,

$$\sigma_{ij}^{aux} = D_{ijkl}(x) \varepsilon_{kl}^{aux}, \varepsilon_{ij}^{aux} \neq \frac{1}{2} (u_{i,j}^{aux} + u_{j,i}^{aux}), \sigma_{ij,j}^{aux} = 0 \quad (6.78)$$

and the constant-constitutive-tensor only satisfies equilibrium and compatibility

$$\sigma_{ij}^{aux} \neq D_{ijkl}(x) \varepsilon_{kl}^{aux}, \varepsilon_{ij}^{aux} = \frac{1}{2} (u_{i,j}^{aux} + u_{j,i}^{aux}), \sigma_{ij,j}^{aux} = 0 \quad (6.79)$$

Table 6.1 briefly compares the three approaches.

Table 6.1 Auxiliary fields for evaluation of interaction integral, reproduced from Kim and Paulino (2003c).

Auxilliary fields	Formulation		
	Non-equilibrium	Incompatible	Constant D_{ijkl}^{tip}
displacement	u^{aux}	u^{aux}	u^{aux}
second variable	$\varepsilon_{ij}^{aux} = \frac{1}{2} (u_{i,j}^{aux} + u_{j,i}^{aux})$	σ_{ij}^{aux}	$\varepsilon_{ij}^{aux} = \frac{1}{2} (u_{i,j}^{aux} + u_{j,i}^{aux})$
third variable	$\sigma_{ij}^{aux} = D_{ijkl}(x) \varepsilon_{kl}^{aux}$	$\varepsilon_{ij}^{aux} = D_{ijkl}^{-1}(x) \sigma_{kl}^{aux}$	$\sigma_{ij}^{aux} = D_{ijkl}^{tip} \varepsilon_{kl}^{aux}$
violated equation	$\sigma_{ij,j}^{aux} \neq 0$	$\varepsilon_{ij}^{aux} \neq \frac{1}{2} (u_{i,j}^{aux} + u_{j,i}^{aux})$	variable $D_{ijkl}(x)$

6.3.3.1 Non-equilibrium formulation

In this method, consistency and constitutive laws are satisfied, while equilibrium is violated. From the constitutive law,

$$\sigma_{ij}^{aux} = D_{ijkl}(x) \varepsilon_{kl}^{aux} \quad (6.80)$$

and its differentiation,

$$\begin{aligned} \sigma_{ij,j}^{aux} &= D_{ijkl,j}(x) \varepsilon_{kl}^{aux} + D_{ijkl}(x) \varepsilon_{kl,j}^{aux} \\ &= [D_{ijkl}^{tip} \varepsilon_{kl,j}^{aux}] + [D_{ijkl,j}(x) \varepsilon_{kl}^{aux}] + [(D_{ijkl}(x) - D_{ijkl}^{tip}) \varepsilon_{kl,j}^{aux}] \end{aligned} \quad (6.81)$$

where the first bracket vanishes. Clearly, it does not satisfy the equilibrium equation $\sigma_{ij,j}^{aux} \neq 0$. Such a choice for the auxiliary field was adopted by Dolbow and Gosz (2002) to solve isotropic FGM problems.

The resultant M integral is then defined as,

$$M^I = \int_A \left\{ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} - \sigma_{ik} \varepsilon_{ik}^{aux} \delta_{1j} \right\} q_{,j} dA + \int_A \left\{ \sigma_{ij,j}^{aux} u_{i,1} - D_{ijkl} \varepsilon_{kl} \varepsilon_{ij}^{aux} \right\} q dA \quad (6.82)$$

where the first term of the second integral ruins the equilibrium due to inhomogeneity of FGM composite material.

6.3.3.2 Incompatible formulation

In this method, equilibrium equations and constitutive laws are satisfied, but the compatibility is violated. The resultant M integral is then defined as,

$$M^I = \int_A \left\{ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} - \sigma_{ik} \varepsilon_{ik}^{aux} \delta_{1j} \right\} q_{,j} dA + \int_A \left\{ \sigma_{ij} (u_{i,1j}^{aux} - \varepsilon_{ij,1}^{aux}) - D_{ijkl} \varepsilon_{kl} \varepsilon_{ij}^{aux} \right\} q dA \quad (6.83)$$

Again, the first term of the second integral, which is due to inhomogeneity of FGM, leads to inconsistency of the formulation.

6.3.3.3 Constant-constitutive-tensor

Finally, the third approach satisfies equilibrium and consistency conditions, but the constitutive laws are not satisfied. The M integral of this formulation can be written as,

$$M^I = \int_A \left\{ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} - \frac{1}{2} (\sigma_{ik} \varepsilon_{ik}^{aux} + \sigma_{ik}^{aux} \varepsilon_{ik}) \delta_{1j} \right\} q_{,j} dA + \int_A \frac{1}{2} \left\{ \sigma_{ij} \varepsilon_{ij,1}^{aux} - \sigma_{ij,1}^{aux} \varepsilon_{ij}^{aux} + \sigma_{ij}^{aux} \varepsilon_{ij,1} - \sigma_{ij,1}^{aux} \varepsilon_{ij} \right\} q dA \quad (6.84)$$

It should be mentioned that in this method, derivatives of real stress and strain fields should be computed. As a result, less accurate results are expected even though more computational efforts are required. On contrary, the first two approaches are faster and easier ways of computing the M integral, and both require similar derivatives of material modulus.

While the constant constitutive tensor formulation needs derivatives of stress and strain in actual fields which leads to inaccuracies in C^0 finite element formulation, the two other cases have more or less the same level of accuracy (Kim and Paulino, 2005).

6.3.3.4 Further simplifications

The incompatibility formulation, which requires less complicated derivatives, is further examined. Beginning with the incompatible formulation (6.83), the following relations (Kim and Paulino, 2003c)

$$\sigma_{ij}^{aux} u_{i,1j} = \sigma_{ik}^{aux} \varepsilon_{ij,1} \quad (6.85)$$

$$\sigma_{ij} \varepsilon_{ij}^{aux} = \sigma_{ik}^{aux} \varepsilon_{ij} \quad (6.86)$$

$$D_{ijkl,1} \varepsilon_{ij}^{aux} \varepsilon_{kl} = D_{ijkl,1} \varepsilon_{ij} \varepsilon_{kl}^{aux} \quad (6.87)$$

can be used to simplify M^I

$$M^I = \int_A \left\{ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} - \sigma_{ik} \varepsilon_{ik}^{aux} \delta_{1j} \right\} q_{,j} dA + \int_A \left\{ \sigma_{ij}^{aux} (u_{i,1j}^{aux} - \varepsilon_{ij,1}^{aux}) - D_{ijkl,1} \varepsilon_{kl} \varepsilon_{ij}^{aux} \right\} q dA \quad (6.88)$$

Evaluation of $\sigma_{ij} \varepsilon_{ij,1}^{aux}$ can be simplified by the method proposed by Kim and Paulino (2004),

$$\begin{aligned} \sigma_{ij} \varepsilon_{ij,1}^{aux} &= \sigma_{ij} \left[C_{ijkl}(x) \sigma_{kl}^{aux} \right]_{,1} \\ &= \sigma_{ij} \left\{ C_{ijkl,1}(x) \sigma_{kl}^{aux} + C_{ijkl}(x) \sigma_{kl,1}^{aux} \right\} \\ &= \sigma_{ij} C_{ijkl,1}(x) \sigma_{kl}^{aux} + \sigma_{ij} \left(C_{ijkl}(x) - C_{ijkl}^{tip} \right) \sigma_{kl,1}^{aux} + \sigma_{ij} \underbrace{C_{ijkl}^{tip} \sigma_{kl,1}^{aux}}_{\varepsilon_{ij,1}^{tip}} \\ &= \sigma_{ij} C_{ijkl,1}(x) \sigma_{kl}^{aux} + \sigma_{ij} \left(C_{ijkl}(x) - C_{ijkl}^{tip} \right) \sigma_{kl,1}^{aux} + \sigma_{ij} u_{i,1j}^{aux} \end{aligned} \quad (6.89)$$

where C_{ijkl} are the components of the compliance tensor, defined in (4.4). Therefore,

$$\sigma_{ij} (u_{i,1j}^{aux} - \varepsilon_{ij,1}^{aux}) = -\sigma_{ij} C_{ijkl,1}(x) \sigma_{kl}^{aux} - \sigma_{ij} \left(C_{ijkl}(x) - C_{ijkl}^{tip} \right) \sigma_{kl,1}^{aux} \quad (6.90)$$

The term $\sigma_{ij} C_{ijkl,1}(x) \sigma_{kl}^{aux}$ vanishes if $r \rightarrow 0$, as a result: (Kim and Paulino, 2004)

$$\left\{ \sigma_{ij} (u_{i,1j}^{aux} - \varepsilon_{ij,1}^{aux}) \right\}_{r \rightarrow 0} \cong -\sigma_{ij} \left(C_{ijkl}(x) - C_{ijkl}^{tip} \right) \sigma_{kl,1}^{aux} \quad (6.91)$$

and

$$M^I = \int_A \left\{ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} - \sigma_{ik} \varepsilon_{ik}^{aux} \delta_{1j} \right\} q_{,j} dA + \int_A \left\{ -\sigma_{ij} \left(C_{ijkl}(x) - C_{ijkl}^{tip} \right) \sigma_{kl,1}^{aux} - D_{ijkl,1} \varepsilon_{kl} \varepsilon_{ij}^{aux} \right\} q dA \quad (6.92)$$

6.3.4 Isoparametric FGM

Any FGM material properties P , such as modules of elasticity E_{11} , E_{22} , shear modules G_{12} and Poisson's ratio ν_{12} vary at different points of the domain in the form of

$$P(x, y) = \varphi(x, y) \quad (6.93)$$

where $\varphi(x, y)$ is a predefined function, usually exponential or linear. Such variable properties lead to variation of constitutive tensors in different points of the inhomogeneous domain. While in the standard FGM modelling, the material properties (6.93) are defined on each Gauss point, in the isoparametric graded finite element method (Kim and Paulino, 2002b), material properties are interpolated from nodal values by FEM shape functions,

$$P(x, y) = \sum_{i=1}^{n_e} N_i(x, y) \bar{P}_i \quad (6.94)$$

where $\bar{P}_i$ is the nodal value of property P . Derivatives of P can then be defined as,

$$P_{,j}(x, y) = \sum_{i=1}^{n_e} N_{,j}(x, y) \bar{P}_i, \quad j = 1, 2 \quad (6.95)$$

Therefore, derivatives of $\mathbf{D}_{ijkl}$ or $\mathbf{C}_{ijkl}$ in the interaction integral (6.82), (6.83) and (6.84) can be computed accordingly.

Also, the roots of equation (4.22) for evaluation of auxiliary fields should be calculated at the crack tip, as described in relation (6.60). Similarly, values of s_{kx} and s_{ky} in enrichment equations (6.104) and (6.105) can also be calculated at the crack tip, but if a large area is to be enriched, s_{kx} and s_{ky} should then be calculated at enriched nodes and interpolated by FEM shape functions (6.94) and (6.95).

6.4. CRACK PROPAGATION IN FGM COMPOSITES

In homogeneous isotropic materials, various crack propagation criteria are available in the literature such as the maximum hoop stress (Erdogan and Sih, 1963), the maximum strain energy release rate (Hussain *et al.*, 1974) and the minimum strain energy density (Sih, 1974). Also, various homogeneous orthotropic mixed mode crack propagation criteria are available such as the maximum circumferential stress by Saouma *et al.* (1987), the minimum strain energy density (Ye and Ayari, 1994) and the maximum circumferential strain criteria (Ayari and Ye, 1995). Unfortunately, little literature is available for crack propagation in inhomogeneous media. Most of the studies in this context have assumed equivalent homogeneous isotropic or orthotropic criteria. In fact, the material behaviour has been idealised as a homogeneous orthotropic continuum to simplify the process of crack modelling. For details see Section 4.4.4.

6.5 INHOMOGENEOUS XFEM

The general methodology of orthotropic extended finite element method (Section 4.5) can be extended to inhomogeneous problems. The crack tip enrichment functions are extracted from the analytical solutions (6.61) and (6.62).

6.5.1 Governing equation

Consider an FGM body subjected to body and traction forces, $\mathbf{f}^b$ and $\mathbf{f}^t$, respectively, as depicted in Fig. 6.2. The strong form of the equilibrium equation in terms of the stress tensor $\boldsymbol{\sigma}$ can be written as:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}^b = 0 \quad \text{in } \Omega \quad (6.96)$$

with the following boundary conditions:

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f}^t \quad \text{on } \Gamma_t : \text{external traction} \quad (6.97)$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u : \text{prescribed displacement} \quad (6.98)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_c : \text{traction free crack} \quad (6.99)$$

where Γ_t , Γ_u and Γ_c are traction, displacement and crack boundaries, respectively.

The variational formulation of the boundary value problem can be defined as:

$$\int_{\Omega} \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \, d\Omega = \int_{\Omega} \mathbf{f}^b \cdot \delta \mathbf{u} \, d\Omega + \int_{\Gamma} \mathbf{f}^t \cdot \delta \mathbf{u} \, d\Gamma \quad (6.100)$$

6.5.2 XFEM approximation

In the extended finite element method, the effects of crack surfaces and crack tips are utilised by the approximation (3.28) to calculate the displacement for a point $\mathbf{x}$ located within the domain (for details see sections 3.4.3):

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{u}^h(\mathbf{x}) + \mathbf{u}^H(\mathbf{x}) + \mathbf{u}^{tip}(\mathbf{x}) + \mathbf{u}^{tra}(\mathbf{x}) \quad (6.101)$$

or more explicitly

$$\begin{aligned} \mathbf{u}^h(\mathbf{x}) = & \left[\sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j \right] + \left[\sum_{h=1}^{mh} N_h(\mathbf{x}) (H(x) - H(x_h)) \mathbf{a}_h \right] \\ & + \left[\sum_{k=1}^{mt} N_k(\mathbf{x}) \left(\sum_{l=1}^{mf} [F_l(x) - F_l(x_k)] \mathbf{b}_k^l \right) \right] \\ & + \left[\sum_{m=1}^{mst} \bar{N}_m(\mathbf{x}) \mathbf{c}_m + \sum_{n=1}^{msh} \bar{N}_n(\mathbf{x}) H(x) \mathbf{d}_n \right] \end{aligned} \quad (6.102)$$

where $\mathbf{u}_j$ is the displacement vector at node j , $\mathbf{b}_k^l$ are the added DOFs for tip enrichments F_l , $\mathbf{a}_h$ are the added DOFs for the crack discontinuity approximation by the Heaviside function H , and $\mathbf{c}_i$ and $\mathbf{d}_j$ are added DOFs which correspond to hierarchical nodes in the sets mst and msh , respectively. N_i are the standard finite element shape functions and $\bar{N}_i$ are the hierarchical shape functions for the transition domain.

6.5.2.1 Orthotropic crack tip enrichment functions

General orthotropic crack tip enrichment functions (4.112) in the local crack tip polar coordinate system (r, θ) are defined as,

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta_1}{2} \sqrt{g_1(\theta)}, \sqrt{r} \cos \frac{\theta_2}{2} \sqrt{g_2(\theta)}, \sqrt{r} \sin \frac{\theta_1}{2} \sqrt{g_1(\theta)}, \sqrt{r} \sin \frac{\theta_2}{2} \sqrt{g_2(\theta)} \right\} \quad (6.103)$$

where functions $g_k(\theta)$ and θ_k , in their general form for all orthotropic composites, are defined as:

$$g_k(\theta) = \sqrt{(\cos \theta + s_{kx} \sin \theta)^2 + (s_{ky} \sin \theta)^2} \quad (6.104)$$

$$\theta_k = \arctan \left(\frac{s_{ky} \sin \theta}{\cos \theta + s_{kx} \sin \theta} \right) \quad (6.105)$$

where s_{kx} and s_{ky} have been defined in Section 4.3.1.

A number of previous studies, such as Dolbow and Gosz (2002), have used the standard (isotropic homogenous) crack tip enrichments to analyse FGMs,

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2} \right\} \quad (6.106)$$

6.5.2.2 Transition domain

The term $\mathbf{u}^{tra}(\mathbf{x})$ has been added to the original XFEM approximation (4.103) to overcome the incompatibility between the enriched and non-enriched domains by introduction of a transition (blending) zone. Two types of hierarchical nodes exist: hierarchical nodes that correspond to edges connecting non-enriched and tip enriched nodes (defined by mst), and those edges that connect tip and Heaviside enriched nodes (defined by mth). Tarancon *et al.* (2009) have discussed the definition of hierarchical shape functions $\bar{N}_i$, as proposed by Szabo and Babuska (1991). For example, for nodes t_1 and t_2 in Fig. 6.3, the hierarchical shape functions in terms of the isoparametric coordinates ξ, η are defined as

$$\bar{N}_{p1}(\xi, \eta) = -\frac{1}{2}\sqrt{\frac{3}{2}}(1-\xi^2)\frac{1-\eta}{2} \quad (6.107)$$

$$\bar{N}_{p2}(\xi, \eta) = -\frac{1}{2}\sqrt{\frac{3}{2}}(1-\eta^2)\frac{1-\xi}{2} \quad (6.108)$$

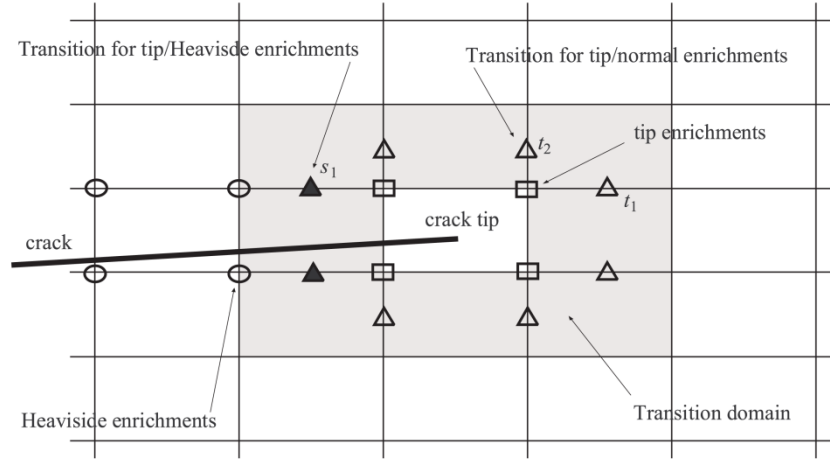


Figure 6.3 Added hierarchical nodes in the blending elements (Tarancon *et al.*, 2009).

6.5.3 XFEM discretisation

Discretisation of Eq. (6.96) using the XFEM approximation (6.101) results in a discrete system of linear equilibrium equations:

$$\mathbf{K}\mathbf{u}^h = \mathbf{f} \quad (6.109)$$

where $\mathbf{u}^h$ is the vector of degrees of nodal freedom (for both classical and enriched ones),

$$\mathbf{u}^h = \{\mathbf{u} \quad \mathbf{a} \quad \mathbf{b}_\alpha \quad \mathbf{c} \quad \mathbf{d}\}^T \quad (6.110)$$

The global stiffness matrix $\mathbf{K}$ is assembled from the stiffness of each element $\mathbf{K}_{ij}^e$ with components $\mathbf{K}_{ij}^{rs}$

$$\mathbf{K}_{ij}^{rs} = \int_{\Omega^e} (\mathbf{B}_i^r)^T \mathbf{D} \mathbf{B}_j^s d\Omega \quad (r, s = \mathbf{u}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \quad (6.111)$$

where $\mathbf{B}_i^u$, $\mathbf{B}_i^a$ and $\mathbf{B}_i^b$ are derivatives of shape functions defined in Eqs. (3.49)-(3.55), and $\mathbf{B}_i^c$ and $\mathbf{B}_i^d$ are defined as,

$$\mathbf{B}_i^c = \begin{bmatrix} \bar{N}_{i,x} & 0 \\ 0 & \bar{N}_{i,y} \\ \bar{N}_{i,y} & \bar{N}_{i,x} \end{bmatrix} \quad (6.112)$$

$$\mathbf{B}_i^d = \begin{bmatrix} (\bar{N}_i H)_{,x} & 0 \\ 0 & (\bar{N}_i H)_{,y} \\ (\bar{N}_i H)_{,y} & (\bar{N}_i H)_{,x} \end{bmatrix} \quad (6.113)$$

$\mathbf{f}$ is the nodal force vector,

$$\mathbf{f}_i^e = \left\{ \mathbf{f}_i^u \quad \mathbf{f}_i^a \quad \mathbf{f}_i^{b^a} \quad \mathbf{f}_i^c \quad \mathbf{f}_i^d \right\}^T \quad (6.114)$$

where $\mathbf{f}_i^u$, $\mathbf{f}_i^a$ and $\mathbf{f}_i^{b^a}$ are defined in (4.109)-(4-111) and the new components are:

$$\mathbf{f}_i^c = \int_{\Gamma_t} \bar{N}_i \mathbf{f}^t \, d\Gamma + \int_{\Omega^e} \bar{N}_i \mathbf{f}^b \, d\Omega \quad (6.115)$$

$$\mathbf{f}_i^d = \int_{\Gamma_t} \bar{N}_i H \mathbf{f}^t \, d\Gamma + \int_{\Omega^e} \bar{N}_i H \mathbf{f}^b \, d\Omega \quad (6.116)$$